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ABSTRACT

Given a graph G on n nodes, let $\mathcal{P}_G$ denote the cone consisting of the positive semidefinite $n \times n$ matrices (with real or complex entries) having a zero entry at every position corresponding to a non edge of G. Then, the order of G is defined as the maximum rank of a matrix lying on an extreme ray of the cone $\mathcal{P}_G$.

It is shown in [AHMR88] that the graphs of order 1 are precisely the chordal graphs and a characterization of the graphs having order 2 is conjectured there in the real case. We show in this paper the validity of this conjecture. Moreover, we characterize the graphs with order 2 in the complex case and we give a decomposition result for the graphs having order ≤ 2 in both real and complex cases. As an application, these graphs can be recognized in polynomial time.

We also establish an inequality relating the order $\mathrm{ord}_{\mathbb{F}}(G)$ of a graph G ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and the parameter fill(G) defined as the minimum number of edges needed to be added to G in order to obtain a chordal graph. Namely, we show that $\mathrm{ord}_{\mathbb{F}}(G) \leq 1 + \epsilon_{\mathbb{F}} \cdot \mathrm{fill}(G)$ where $\epsilon_{\mathbb{R}} = 1$ and $\epsilon_{\mathbb{C}} = 2$; this settles a conjecture posed in [HPR89].

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1 Introduction

In this paper we study the ranks of extremal positive semidefinite matrices with a given sparsity pattern, in continuation of the papers [AHMR88], [HPR89], [McC88, McC93], [HLW94]. This study is motivated mainly by its application to the completion problem for positive semidefinite matrices (details are given below) and it is also relevant to chordal graphs and Gaussian elimination for sparse positive definite matrices ([Ro70]).

The order of a graph. Given a graph $G = (V, E)$ with node set $V = \{1, 2, \ldots, n\}$, let $\mathcal{P}_G$ denote the set of positive semidefinite $n \times n$ matrices whose ij-th entry is zero for every $i \neq j \in V$ such that edge (i, j) does not belong to E. All matrices are assumed to have entries in the field $\mathbb{F}$, where $\mathbb{F}$ is equal to $\mathbb{R}$ (the field of real numbers) or $\mathbb{C}$ (the field of complex numbers), and matrices in $\mathcal{P}_G$ are assumed to be Hermitian.

The set $\mathcal{P}_G$ is a closed convex cone. A matrix $X \in \mathcal{P}_G$ is said to be *extremal* if it lies on an extreme ray of the cone $\mathcal{P}_G$. When G is the complete graph K_n (i.e., no entries have prescribed zeros), then $\mathcal{P}_{K_n}$ is the familiar cone of positive semidefinite matrices and, as is well-known, all extremal matrices in $\mathcal{P}_{K_n}$ have rank ≤ 1. For a graph G, its *order* $\mathrm{ord}_{\mathbb{F}}(G)$ is defined as the maximum rank of an extremal matrix in $\mathcal{P}_G$; clearly, $\mathrm{ord}_{\mathbb{F}}(G) \leq n-1$ for a graph G on n nodes (and, in the real case, $\mathrm{ord}_{\mathbb{R}}(G) \leq n-2$ if $n \geq 3$).

A question of interest is to characterize the graphs whose order is less than or equal to a prescribed value k. Such characterization is known in the case when $k = 1$.

Theorem 1. [AHMR88] *For a graph G, we have:* $\mathrm{ord}_{\mathbb{R}}(G) = 1 \Longleftrightarrow \mathrm{ord}_{\mathbb{C}}(G) = 1$ $\Longleftrightarrow$ *G is chordal, i.e., does not contain any circuit of length ≥ 4 as an induced subgraph.* ∎

Non chordal graphs may have an arbitrarily large order. Indeed, a circuit of length $k+2$ has order $\geq k$. Thus, there seems to be some link between the order of a graph and its 'deficiency' in being chordal. For a graph G, let $\mathrm{fill}(G)$ denote the minimum number of edges that need to be added to G in order to obtain a chordal graph. The parameter $\mathrm{fill}(G)$ is called the *minimum fill-in* of G; it has been studied, in particular, in connection with the Gaussian elimination process for real symmetric positive definite matrices (cf. Rose [Ro70]). Thus, $\mathrm{fill}(G) = 0$ when G is chordal and $\mathrm{fill}(G) = k-1$ if G is a circuit of length $k+2$. The following inequalities relating the order of a graph G and its minimum fill-in are conjectured by Helton, Pierce and Rodman [HPR89]:

$$\mathrm{ord}_{\mathbb{R}}(G) \leq \mathrm{fill}(G) + 1, \quad \mathrm{ord}_{\mathbb{C}}(G) \leq 2 \cdot \mathrm{fill}(G) + 1.$$

We will show that these inequalities indeed hold (cf. Theorem 7). One can easily verify that

$$\mathrm{ord}_{\mathbb{F}}(H) \leq \mathrm{ord}_{\mathbb{F}}(G) \text{ if } H \text{ an induced subgraph of } G.$$

Agler et al. [AHMR88] have shown that, if a graph has order ≤ 2 over $\mathbb{R}$, then it does not contain as an induced subgraph a circuit of length ≥ 5 or any of the sixteen graphs shown in Figure 2 (all having order 3). They conjectured that this implication holds, in fact, as an equivalence. We show in this paper the validity of this conjecture (cf. Theorem 10) and we prove an analogous characterization for the graphs having order ≤ 2 over $\mathbb{C}$ (cf. Theorem 15). In both the real and complex cases, we give a decomposition result for the graphs of order ≤ 2 as clique sums of some basic classes of graphs. As an application, one can recognize in polynomial time whether a graph has order 1 or 2. As another application, we can characterize the graphs whose 'powers' all have order ≤ 2 (cf. Theorems 12 and 17); in the complex case we find again a result of McCullough [McC88, McC93]. Moreover, we obtain the classification of the 3-blocks (the minimal - with respect to taking induced subgraphs - graphs of order 3), which was not known in the complex case.

Let us summarize here the contents of the paper. In Section 2, we prove our first result, relating the order of a graph with its minimum fill-in. Section 3 is devoted to the characterization of the graphs having order ≤ 2 over the reals and Section 4 solves the same problem in the complex case. Our results are based essentially on a graph-theoretic result providing a decomposition scheme for a class of graphs defined by excluding certain graphs as induced subgraphs (cf. Theorem 9); Section 5 is devoted to the proof of this result which is quite technical.

Application to the completion problem. Let us now explain the link existing between the study of the cone $\mathcal{P}_G$ and of the order of a graph and the completion problem for positive semidefinite matrices.

The matrix completion problem consists of deciding whether a given partial matrix can be completed so as to obtain a matrix satisfying a prescribed matrix property, in our case, being positive semidefinite. This problem has received a lot of attention in the literature; this is due, in particular, to its many applications, e.g., to statistics, molecular chemistry, distance geometry, etc. (Cf. the surveys by Johnson [Jo90], Laurent [La97] and further references there.)

To be more precise, a *partial matrix* $A = (a_{ij})$ of order n is a matrix whose entries are specified only on a subset of the positions including all diagonal positions; hence, we may represent the set of off-diagonal positions at which the entries are specified as the edge set of a graph $G = (V, E)$ where $V = \{1, \ldots, n\}$. The partial matrix A is said to be *completable to a postive semidefinite matrix* if there exist values b_{ij} ($i \neq j \in V$, $ij \notin E$) such that the matrix with entry a_{ij} if $i = j$ or $i \neq j$ with $ij \in E$ and with entry b_{ij} if $i \neq j$ with $ij \notin E$, is positive semidefinite. Then, we let $\mathcal{C}_G$ denote the set of all such completable partial matrices A. Thus, $\mathcal{C}_G$ is a cone in the space $\mathbb{R}^{V \cup E}$. Roughly speaking, the cone $\mathcal{C}_G$ and the cone $\mathcal{P}_G$ are polar cones.

Recall that the *polar* C° of a cone $C \subseteq \mathbb{F}^d$ is the set of all $y \in \mathbb{F}^d$ such that $y^* x \geq 0 \; \forall x \in C$. When the cone C consists of $n \times n$ matrices, we view C as a subset of $\mathbb{F}^{n^2}$ equipped with the usual inner product. That is, for two $n \times n$ matrices A and B, their inner product $\langle A, B \rangle$ is defined as $\mathrm{Tr}(A^* B) = \sum_{i,j=1}^n a_{ij}^* b_{ij}$. (Here, z^*, a^*, A^* denote, respectively, the conjugate of $z \in \mathbb{F}$ (equal to z if $\mathbb{F} = \mathbb{R}$), and the conjugate transpose of vector a or matrix A.)

When G is the complete graph K_n, $\mathcal{C}_{K_n}$ is nothing but the cone of $n \times n$ positive semidefinite matrices, which coincides with its polar cone. More generally, one can easily verify that

(1) $\mathcal{P}_G = (\mathcal{C}_G)^\circ.$

(To be precise, the polar cone of $\mathcal{C}_G$ coincides with the projection of the cone $\mathcal{P}_G$ on the subspace $\mathbb{R}^{V \cup E}$.) Therefore, a partial matrix is completable to a positive semidefinite matrix if and only if its inner product with any extremal matrix in $\mathcal{P}_G$ is nonnegative. Hence, knowledge about the extremal matrices in $\mathcal{P}_G$ is useful for deciding completability of partial matrices. This fact motivates the study of extremal matrices in $\mathcal{P}_G$ and of the order of graph G. In view of relation (1), Theorem 1 can be easily derived from the following result of [GJSW84] concerning the cone $\mathcal{C}_G$.

Theorem 2. *A graph G is chordal if and only if every partial matrix whose entries are specified on G (and on the main diagonal) and for which all fully specified principal submatrices are positive semidefinite can be completed to a positive semidefinite matrix.* ∎

In the rest of the section, we give some notation on graphs and matrices, as well as some preliminaries on the order of a graph and the related notion of k-block.

Graphs. Although our graph terminology is standard, we yet recall some definitions that are used in the paper. All graphs are assumed here to be simple (i.e., without loops and parallel edges). Let $G = (V, E)$ be a graph. Its *complementary graph* is the graph $\overline{G} := (V, \overline{E})$ where $\overline{E}$ consists of the pairs $ij \notin E$ with $i \neq j \in V$. G is called a *clique* (or *complete graph*) if $ij \in E$ for all $i \neq j \in V$. Then, K_n denotes the clique on n nodes. As usual, $K_{n,m}$ denotes the complete bipartite graph with colour classes of cardinalities n and m, and C_n denotes the circuit of length n (the circuit $C_n = (i_1, \ldots, i_n)$ has node set $\{i_1, \ldots, i_n\}$ and edges $i_1 i_n$ and $i_j i_{j+1}$ for $j = 1, \ldots, n-1$). A path of length n has the form $(i_1, \ldots, i_{n+1})$; its node set is $\{i_1, \ldots, i_{n+1}\}$ and its edges are the pairs $i_j i_{j+1}$ for $j = 1, \ldots, n$. Given a subset U of V, $G[U]$ denotes the *subgraph of G induced by U*; its node set is U and its edge set is $\{ij \in E \mid i, j \in U\}$. A subset $S \subseteq V$ is called a *stable set* of G if $ij \notin E$ for all $i \neq j \in S$. Then, $\alpha(G)$ denotes the *stability number* of G, defined as the maximum cardinality of a stable set in G. A subset $F \subseteq E$ is called a *matching* in G if no two edges of F have a common endnode.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that the set $K := V_1 \cap V_2$ is a clique in both G_1, G_2 and there is no edge between a node of $V_1 \setminus V_2$ and a node of $V_2 \setminus V_1$. Then, the graph $G := (V_1 \cup V_2, E_1 \cup E_2)$ is called the *clique sum* of the graphs G_1 and G_2 and the set K is called a *clique cutset* of G.

A graph G is said to be *chordal* (or *triangulated*) if it does not contain a circuit C_n ($n \geq 4$) as an induced subgraph. Equivalently, G is chordal if and only if G is a clique sum of cliques (Dirac [Di61]).

Given an integer $m \geq 1$, let $G^{(m)}$ denote the graph obtained from G by replacing every node $v \in V$ by a clique K_v of cardinality m and making any two nodes $i \in K_u$, $j \in K_v$ adjacent in $G^{(m)}$ if and only if the nodes u and v are adjacent in G.

Matrices and vector representations. An $n \times n$ matrix X with entries in $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) is *Hermitian* if $X^* = X$ holds (thus, Hermitian means symmetric in the real case). A Hermitian matrix $X = (x_{ij})$ is said to be *positive semidefinite* (then, we write: $X \succeq 0$) if $x^* X x \geq 0$ for all $x \in \mathbb{F}^n$. Equivalently, $X \succeq 0$ if there exist vectors $u_1, \ldots, u_n \in \mathbb{F}^k$ ($k \geq 1$) such that $x_{ij} = u_i^* u_j$ for all $i, j = 1, \ldots, n$; the set of vectors $(u_1, \ldots, u_n)$ is then called a *Gram representation* of X and X is called their *Gram matrix*. Note that $\{u_1, \ldots, u_n\}$ and X have the same rank. If X has rank k, then it has a unique (up to orthogonal transformation) Gram representation in the k-dimensional space $\mathbb{F}^k$.

Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$ and with complementary graph $\overline{G} = (V, \overline{E})$. For every matrix $X \in \mathcal{P}_G$ with Gram representation $u_1, \ldots, u_n$, the vectors $u_1, \ldots, u_n$ satisfy the relation:

$$u_i^* u_j = 0 \ \text{ for all } ij \in \overline{E}.$$

An assignment of vectors to the nodes of G satisfying the above condition is called an *orthogonal representation* of G. The notion of order of a graph G can, therefore, be reformulated in terms of orthogonal representations of G (cf. Definition 6). Orthogonal representations were introduced by Lovász [Lo79] in the study of the Shannon capacity of a graph; they arise in connection with various topics in combinatorial optimization and discrete geometry, like the vertex packing polytope of a graph (cf. [GLS86, GLS88]), connectivity properties of graphs (cf. [LSS89]), spectral invariants of graphs (cf. [CdV], [vdH96]).

k-Blocks. Following [AHMR88], a graph G is called a *k-block* if G has order k and every proper induced subgraph of G has order $\leq k-1$. For instance, the circuit C_n is an $(n-2)$-block over the reals if $n \geq 4$ [AHMR88]. To see it, consider the following vectors $u_i \in \mathbb{R}^{n-2}$:

$$\begin{array}{l} u_1 := e_1, \quad u_i := e_{i-1} + e_i \ \text{ for } i = 2, \ldots, n-2, \\ u_{n-1} := e_{n-2}, \quad u_n := \sum_{i=1}^{n-2} (-1)^i e_i. \end{array}$$

where $e_1, \ldots, e_{n-2}$ denote the unit vectors in $\mathbb{R}^{n-2}$. Their Gram matrix is an extremal matrix of $\mathcal{P}_{C_n}$ and has rank $n-2$. Therefore, $\mathrm{ord}_{\mathbb{R}}(C_n) \geq n-2$ and equality holds since $\mathrm{ord}_{\mathbb{R}}(G) \leq n-2$ for a graph on n nodes ([AHMR88]; this follows easily using relation (4)). Therefore, C_n is an $(n-2)$-block over $\mathbb{R}$, because every proper induced subgraph of C_n is chordal and, thus, has order 1. The above assignment of vectors to the nodes of C_n shows that $\mathrm{ord}_{\mathbb{C}}(C_n) \geq n-2$; but $\mathrm{ord}_{\mathbb{C}}(C_n) \leq n-2$ (using (4)) and, thus, C_n is also an $(n-2)$-block over $\mathbb{C}$.

Obviously, a graph G has order $\leq k$ if and only if G does not contain as an induced subgraph a p-block with $p > k$. Therefore, a characterization of the graphs with order $\leq k$ would follow if one would know a classification of the p-blocks for any $p > k$. Such a classification has been obtained in [AHMR88] for the p-blocks over $\mathbb{R}$ when $p \leq 3$. It is shown there that, over the reals, K_1 is the only 1-block, C_4 is the only 2-block, and there are exactly sixteen 3-blocks (cf. Theorem 11).

The number of nodes of a p-block is at most p^2+p-2 if $\mathbb{F} = \mathbb{R}$ and at most p^2-1 if $\mathbb{F} = \mathbb{C}$ ([AHMR88]). Hence, there is a finite number of p-blocks. Yet classifying p-blocks seems to be quite hard, already for $p = 4$. Indeed, Helton, Lam and Woederman [HLW94] have classified the 4-blocks over $\mathbb{R}$ having minimum number 9 of nonedges (indeed, by relation (4), a graph of order 4 has at least $\binom{5}{2} - 1 = 9$ nonedges); their number is quite large and their classification involves many technical details, which indicates the difficulty of treating the general case.

However, in order to characterize the graphs having order $\leq k$, one needs only to know the minimal (with respect to taking induced subgraphs) graphs among the p-blocks with $p > k$ and this might be more tractable, at least for small values of k. For instance, we deduce from Theorem 1 that every p-block ($p \geq 2$) contains some circuit of length ≥ 4 as an induced subgraph. The following is conjectured in [AHMR88] in the case $k = 2$ and $\mathbb{F} = \mathbb{R}$:

Conjecture 3. *A graph G satisfies:* $\mathrm{ord}_{\mathbb{R}}(G) \leq 2$, *if and only if G does not contain as an induced subgraph a circuit on $n \geq 5$ nodes or a 3-block. Equivalently, over the reals, the only k-block ($k \geq 4$) which contains no 3-block is the circuit C_{k+2}.*

The main contribution of this paper is to show the validity of Conjecture 3. The essential ingredient in our proof consists of giving a decomposition result for the class of graphs having no 3-block and no circuit of length ≥ 5 as an induced subgraph (cf. Theorem 9). This decomposition result involves clique sums of graphs and we will use the following result of Helton, Pierce and Rodman [HPR89].

Proposition 4. [HPR89] *If G is the clique sum of two graphs G_1 and G_2, then*

$$\mathrm{ord}_{\mathbb{F}}(G) = \max(\mathrm{ord}_{\mathbb{F}}(G_1), \mathrm{ord}_{\mathbb{F}}(G_2)).$$

Note that it is very easy to see that chordal graphs have order 1 using Proposition 4; indeed, any clique has order 1 and chordal graphs are clique sums of cliques.

2 Relating the order of a graph and its deficiency in chordality

In this section we establish an inequality relating the order of a graph over $\mathbb{R}$ or $\mathbb{C}$ and its minimum fill-in. This result will be essentially based on a formula permitting to express the dimension of a face of the cone $\mathcal{P}_G$ in terms of parameters of matrices in this face. We begin with establishing the latter result about face dimensions.

Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$. A subset $F \subseteq \mathcal{P}_G$ is called a *face* of the cone $\mathcal{P}_G$ if $X = Y + Z$ with $X \in F$, $Y, Z \in \mathcal{P}_G$ implies that $Y, Z \in F$. Then, the *extreme rays* of $\mathcal{P}_G$ are its faces of dimension 1 and a matrix $X \in \mathcal{P}_G$ is said to be *extremal* if X lies on an extreme ray. Given $X \in \mathcal{P}_G$, let $F_{\mathcal{P}_G}(X)$ denote the smallest (with respect to inclusion) face of $\mathcal{P}_G$ that contains X. We have:

$$F_{\mathcal{P}_G}(X) = \{Y \in \mathcal{P}_G \mid \mathrm{Ker} X \subseteq \mathrm{Ker} Y\}$$

(where $\mathrm{Ker}X = \{x \in \mathbb{R}^n \mid Xx = 0\}$); this relation was shown in [HW87] in the case when $G = K_n$ and the general case follows easily. Moreover, one can compute the dimension of the face $F_{\mathcal{P}_G}(X)$ in terms of parameters of X.

We introduce some notation. Let $\overline{G} = (V, \overline{E})$ denote the complementary graph of G. Let $X \in \mathcal{P}_G$ have rank k and with Gram representation $u_1, \ldots, u_n \in \mathbb{F}^k$. Given a subset $A \subseteq E \cup \overline{E}$, we introduce the following set U_A of $k \times k$ matrices:

$$\textbf{(2)} \qquad \begin{aligned} &U_A := \{u_i u_j^* + u_j u_i^* \mid ij \in A\} && \text{if } \mathbb{F} = \mathbb{R}, \\ &U_A := \{u_i u_j^*, u_j u_i^* \mid ij \in A\} && \text{if } \mathbb{F} = \mathbb{C}. \end{aligned}$$

Note that, when $A = \overline{E}$, all matrices in $U_{\overline{E}}$ are orthogonal to the identity matrix; thus, the rank of $U_{\overline{E}}$ is less than or equal to $\binom{k+1}{2} - 1$ ($\mathbb{F} = \mathbb{R}$) or $k^2 - 1$ ($\mathbb{F} = \mathbb{C}$). As shown in [AHMR88] and as follows from the next result, equality characterizes extremality of X. Theorem 5 is an analogue of a result of Li and Tam [LT94] who considered instead of $\mathcal{P}_G$ the cone of positive semidefinite matrices with diagonal entries one. In fact, a generalization of Theorem 5 holds where, instead of $\mathcal{P}_G$, one considers the cone $\mathcal{P}_{K_n}$ intersected by a finite number of arbitrary hyperplanes (cf. Theorem 31.5.3 in [DL97]).

Theorem 5. *Let $G = (V, E)$ be a graph with complementary graph $\overline{G} = (V, \overline{E})$, let $X \in \mathcal{P}_G$ have rank k, let $u_1, \ldots, u_n \in \mathbb{R}^k$ be a Gram representation of X, and let $U_{\overline{E}}$ be defined by* (2)*. Then,*

$$\textbf{(3)} \qquad \dim F_{\mathcal{P}_G}(X) = \binom{k+1}{2} - \mathrm{rank}_{\mathbb{R}}(U_{\overline{E}}) \ (\mathbb{F} = \mathbb{R}), \ \ k^2 - \mathrm{rank}_{\mathbb{C}}(U_{\overline{E}}) \ (\mathbb{F} = \mathbb{C}).$$

In particular, X is extremal if and only if $\mathrm{rank}_{\mathbb{F}}(U_{\overline{E}}) = \binom{k+1}{2} - 1$ *($\mathbb{F} = \mathbb{R}$) or $k^2 - 1$ ($\mathbb{F} = \mathbb{C}$).*

Proof. Call a $k \times k$ matrix B a *perturbation* of X if $X \pm \lambda B \in \mathcal{P}_G$ for some $\lambda > 0$ and let $\mathcal{B}$ denote the set of perturbations of X. Then, dim $F_{\mathcal{P}_G}(X)$ is clearly equal to the dimension of the set $\mathcal{B}$. Let U denote the $k \times n$ matrix whose columns are the vectors $u_1, \ldots, u_n$. Then, $X = U^* U$. We claim:

$$\text{(a)} \qquad B \in \mathcal{B} \iff \begin{array}{l} B = U^*RU \text{ for some } k \times k \text{ Hermitian matrix } R \\ \text{satisfying: } \langle R, u_i u_j^* \rangle = \langle R, u_j u_i^* \rangle = 0 \text{ for all } ij \in \overline{E}. \end{array}$$

Note first that the condition: $\langle R, u_i u_j^* \rangle = \langle R, u_j u_i^* \rangle = 0 \ \forall ij \in \overline{E}$, ensures that the matrix $X \pm \lambda B$ has zero entries at the positions corresponding to non edges of G. We now verify that $X \pm \lambda B \succeq 0$ for some $\lambda > 0$ if and only if $B = U^*RU$ for some $k \times k$ Hermitian matrix R. The 'only if' part is clear. Suppose now that $X \pm \lambda B \succeq 0$ for some $\lambda > 0$. Complete U to an $n \times n$ nonsingular matrix V with entries in $\mathbb{F}$. Set $C := (V^{-1})^* B V^{-1}$; that is, $B = V^* C V$. Note that B and, thus, C are Hermitian. For $\epsilon = \pm 1$, we have:

$$X + \epsilon\lambda B = V^* \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} V + \epsilon\lambda V^* C V = V^* \begin{pmatrix} I_k + \epsilon\lambda C_1 & \epsilon\lambda C_0 \\ \epsilon\lambda C_0^* & \epsilon\lambda C_2 \end{pmatrix} V$$

after setting $C := \begin{pmatrix} C_1 & C_0 \\ C_0^* & C_2 \end{pmatrix}$. As $X + \epsilon\lambda B \succeq 0$ for $\epsilon = \pm 1$ (with $\lambda > 0$), this implies that $C_2 = C_0 = 0$. Therefore, we obtain that

$$B = V^* \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} V = U^* C_1 U,$$

where C_1 is $k \times k$ Hermitian. Thus, (a) holds.

We can now derive relation (3). Let $\mathcal{U}$ denote the subspace of $\mathbb{F}^{k^2}$ (the set of $k \times k$ matrices) spanned by $U_{\overline{E}}$. Then, by (a), $\dim \mathcal{B}$ is equal in the real case to the dimension of the set $\mathcal{S}_k \cap \mathcal{U}^\perp$ (the orthogonal complement of $\mathcal{U}$ in the space $\mathcal{S}_k$ of real symmetric matrices) and, thus, to $\binom{k+1}{2} - \text{rank}_{\mathbb{R}}(U_{\overline{E}})$ (we have used here the fact that, for a symmetric matrix B, $\langle B, u_i u_j^* \rangle = 0 \iff \langle B, u_i u_j^* + u_j u_i^* \rangle = 0$). In the complex case, $\dim \mathcal{B}$ is equal to the dimension of the set $\mathcal{H}_k \cap \mathcal{U}^\perp$ (where $\mathcal{H}_k$ denotes the set of $k \times k$ complex Hermitian matrices). We observe that this set has the same dimension as its superset $\mathbb{C}^{k^2} \cap \mathcal{U}^\perp$, which implies that $\dim \mathcal{B} = k^2 - \text{rank}_{\mathbb{C}}(U_{\overline{E}})$. (Indeed, suppose that $\{R_1, \ldots, R_p\}$ is a set of linearly independent matrices in $\mathbb{C}^{k^2} \cap \mathcal{U}^\perp$. Note that for $R \in \mathbb{C}^{k^2} \cap \mathcal{U}^\perp$, both matrices $R + R^*$ and $i(R - R^*)$ belong to $\mathcal{H}_k \cap \mathcal{U}^\perp$. Moreover, at least one of the two systems $\{R_1 + R_1^*, R_2, \ldots, R_p\}$ and $\{i(R_1 - R_1^*), R_2, \ldots, R_p\}$ is linearly independent. Therefore, we can iteratively construct from $\{R_1, \ldots, R_p\}$ a set of p linearly independent matrices in $\mathcal{H}_k \cap \mathcal{U}^\perp$.) ∎

In view of Theorem 5, the order of a graph can be equivalently defined as follows.

Definition 6. *Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$ and complementary graph $\overline{G} = (V, \overline{E})$. Its* order $\mathrm{ord}_{\mathbb{F}}(G)$ *is defined as the largest integer k for which there exists a set of vectors $u_1, \ldots, u_n \in \mathbb{F}^k$ having rank k and satisfying:*

$$u_i^T u_j = 0 \ \forall ij \in \overline{E}, \ \mathrm{rank}(U_{\overline{E}}) = \binom{k+1}{2} - 1 (\mathbb{F} = \mathbb{R}), \ k^2 - 1 (\mathbb{F} = \mathbb{C}).$$

Such a set of vectors is called a k-dimensional extremal orthogonal representation *of G.*

Therefore, a graph G of order k must have sufficiently many non edges; namely,

$$\textbf{(4)} \qquad |\overline{E}| \geq \binom{k+1}{2} - 1 \ (\mathbb{F} = \mathbb{R}), \frac{1}{2}(k^2 - 1) \ (\mathbb{F} = \mathbb{C}).$$

Theorem 7. *For any graph G, we have:*

$$\mathrm{ord}_{\mathbb{R}}(G) \leq \mathrm{fill}(G) + 1, \ \mathrm{ord}_{\mathbb{C}}(G) \leq 2 \cdot \mathrm{fill}(G) + 1.$$

Proof. Let $G = (V, E)$ be a graph and let $\overline{G} = (V, \overline{E})$ denote its complementary graph. Set $k := \mathrm{ord}_{\mathbb{F}}(G)$ and $p := \mathrm{fill}(G)$. There exists a subset F of $\overline{E}$ of cardinality p such that the graph $H := (V, E \cup F)$ is chordal. Let X be an extremal matrix in $\mathcal{P}_G$ of rank k and with Gram representation $u_1, \ldots, u_n \in \mathbb{F}^k$; thus, $X \in \mathcal{P}_H$. Set $\rho_{\mathbb{R}} := \binom{k+1}{2}$ and $\rho_{\mathbb{C}} := k^2$. By relation (3), we have:

$$\mathrm{rank}_{\mathbb{F}}(U_{\overline{E}}) = \rho_{\mathbb{F}} - \dim F_{\mathcal{P}_G}(X) = \rho_{\mathbb{F}} - 1,$$

$$\mathrm{rank}_{\mathbb{F}}(U_{\overline{E} \setminus F}) = \rho_{\mathbb{F}} - \dim F_{\mathcal{P}_H}(X).$$

On the other hand,

$$\mathrm{rank}_{\mathbb{F}}(U_{\overline{E}}) \leq \mathrm{rank}_{\mathbb{F}}(U_{\overline{E} \setminus F}) + \mathrm{rank}_{\mathbb{F}}(U_F) \leq \mathrm{rank}_{\mathbb{F}}(U_{\overline{E} \setminus F}) + \epsilon_{\mathbb{F}} \cdot |F|,$$

setting $\epsilon_{\mathbb{R}} = 1$ and $\epsilon_{\mathbb{C}} := 2$. This implies that

$$\dim F_{\mathcal{P}_H}(X) \leq \epsilon_{\mathbb{F}} \cdot |F| + 1 = \epsilon_{\mathbb{F}} \cdot p + 1.$$

There exist $d \leq \dim F_{\mathcal{P}_H}(X)$ extremal matrices $X_1, \ldots, X_d \in \mathcal{P}_H$ such that $X = X_1 + \ldots + X_d$. This implies that $\mathrm{rank} X \leq \mathrm{rank} X_1 + \ldots + \mathrm{rank} X_d$. Each matrix X_i has rank 1 since H is chordal. Therefore, $\mathrm{rank} X \leq d$ which, combined with the inequality: $d \leq \epsilon_{\mathbb{F}} \cdot p + 1$, implies that $k = \mathrm{rank} X \leq \epsilon_{\mathbb{F}} \cdot p + 1$. That is, $\mathrm{ord}_{\mathbb{F}}(G) \leq \epsilon_{\mathbb{F}} \cdot \mathrm{fill}(G) + 1$. ∎

The inequality in Theorem 7 is tight when $\mathrm{ord}_{\mathbb{F}}(G) = 1$ but it is not tight in general. The difference between the minimum fill-in and the order can, in fact, be arbitrarily large. Indeed, if G is the clique sum of two graphs G_1 and G_2, then $\mathrm{ord}_{\mathbb{F}}(G) = \max(\mathrm{ord}_{\mathbb{F}}(G_1), \mathrm{ord}_{\mathbb{F}}(G)_2)$ while $\mathrm{fill}(G) = \mathrm{fill}(G_1) + \mathrm{fill}(G_2)$. We will see in Section 3 examples of graphs (those in class $\mathcal{G}_4$ - they are not clique sums) having order 2 and an arbitrarily large minimum fill-in.

The complexity of computing the order of a graph is not known. On the other hand, evaluating the upper bound given by Theorem 7 is hard, since computing the minimum fill-in is NP-complete ([Ya81]).

3 Graphs of order 2: The real case

In this section, we characterize the graphs having order ≤ 2 in the real case. The main result is Theorem 10 which gives two equivalent descriptions for these graphs; one is in terms of forbidden induced subgraphs and the other one shows how such graphs can be decomposed by means of clique sums using four basic classes of graphs. In particular, this result shows validity of Conjecture 3, which was posed by Agler et al. [AHMR88].

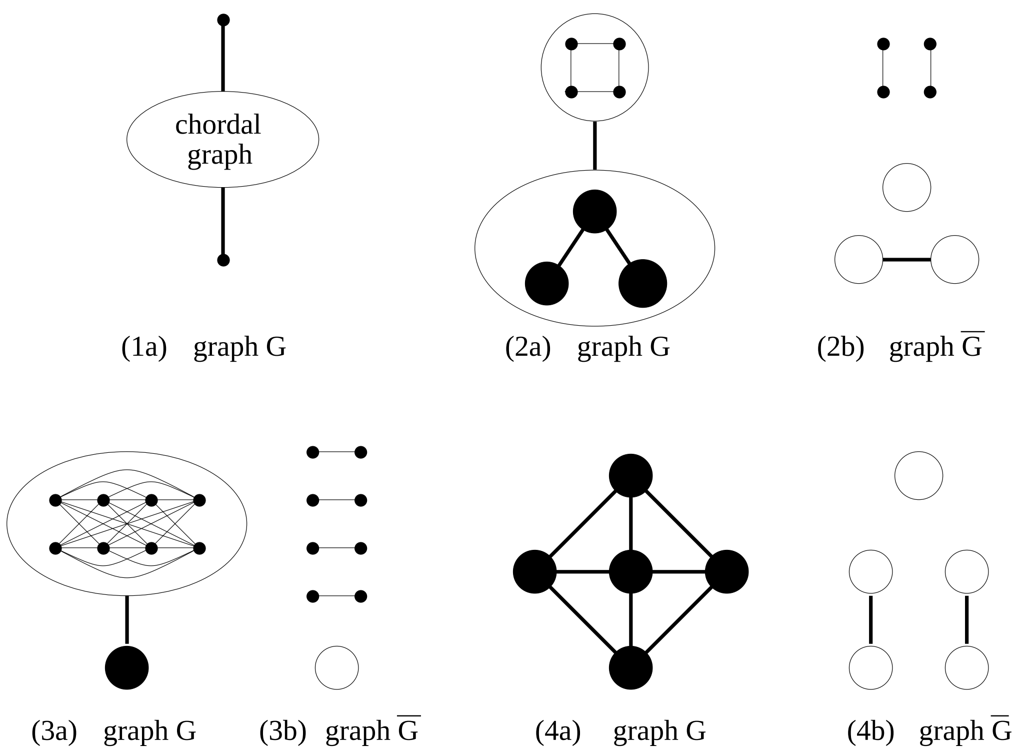

Figure 1: Classes $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$

We begin with introducing four classes of graphs having order ≤ 2 over $\mathbb{R}$. These graphs are shown in Figure 1. For $i = 1, 2, 3, 4$, let $\mathcal{G}_i$ denote the class consisting of the graphs having the form shown in Figure 1 (ia) and of their induced subgraphs.

For each class $\mathcal{G}_i$ with $i = 2, 3, 4$, we picture not only graph $G \in \mathcal{G}_i$ but also its complementary graph $\overline{G}$, because the latter graph has a very simple form which will be used in the proof of Proposition 8. Note that a graph $G \in \mathcal{G}_1$ is obtained by adding two non adjacent nodes to a chordal graph H and making them adjacent to all nodes of H (and taking an induced subgraph of the resulting graph).

We use the following convention in Figure 1: A small dark dot indicates a node, a big dark sphere indicates a clique, while a big white sphere indicates a stable set; edges are indicated by lines, while a thick line between two spheres or between two sets of nodes shows that every node in one set is adjacent to every node in the other set.

Remark that a graph in class $\mathcal{G}_i$ has minimum fill-in at most i for $i = 1, 2, 3$; on the other hand, graphs in $\mathcal{G}_4$ may have an arbitrary large minimum fill-in.

Proposition 8. *We have:* $\mathrm{ord}_{\mathbb{R}}(G) \leq 2$ *for* $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$.

PROOF. If $G \in \mathcal{G}_1$, then $\mathrm{ord}_{\mathbb{R}}(G) \leq 2$ follows from Theorem 7, since $\mathrm{fill}(G) \leq 1$. Let $G \in \mathcal{G}_i$ for $i = 2, 3, 4$. Let X be an extremal matrix in the cone $\mathcal{P}_G$ having rank $k := \mathrm{ord}_{\mathbb{R}}(G)$ and with Gram representation $u_1, \ldots, u_n \in \mathbb{R}^k$. Then, by Theorem 5,

$$\mathrm{rank}(U_{\overline{E}}) = \frac{1}{2}(k^2 + k - 2).$$

We compute in each case the rank of the set $U_{\overline{E}}$. Consider first the case when $G \in \mathcal{G}_2$. Let A, B denote the node sets corresponding to the two stable sets that are connected in $\overline{G}$ (cf. Figure 1 (2b)) and set $a := \mathrm{rank}\{u_i \mid i \in A\}$, $b := \mathrm{rank}\{u_i \mid i \in B\}$. Then,

$$\mathrm{rank}(U_{\overline{E}}) \leq 2 + ab.$$

We have that $a + b \leq k$ since every u_i $(i \in A)$ is orthogonal to every u_j $(j \in B)$; this implies that $ab \leq \frac{1}{4}k^2$. Therefore, we have:

$$\frac{1}{2}(k^2 + k - 2) \leq 2 + \frac{1}{4}k^2,$$

from which follows that $k \leq 2$. If $G \in \mathcal{G}_3$, then we have:

$$\frac{1}{2}(k^2 + k - 2) \leq 4,$$

implying again that $k \leq 2$. Finally, if $G \in \mathcal{G}_4$, then we obtain in the same way as above that:

$$\frac{1}{2}(k^2 + k - 2) \leq \frac{1}{4}k^2 + \frac{1}{4}k^2,$$

which also implies that $k \leq 2$. ∎

We show in Figure 2 the complementary graphs of $A_1 - A_{10}$, $B_1 - B_6$, since they have a simpler form. Note that A_1, B_2, A_2, B_4, B_5 are, respectively, the circuit C_5, the complete bipartite graph $K_{3,3}$, $K_{3,3} + e$ (add an edge to $K_{3,3}$), $K_{3,3} \backslash f$ (delete an edge from $K_{3,3}$), $K_{3,3} + e \backslash f$.

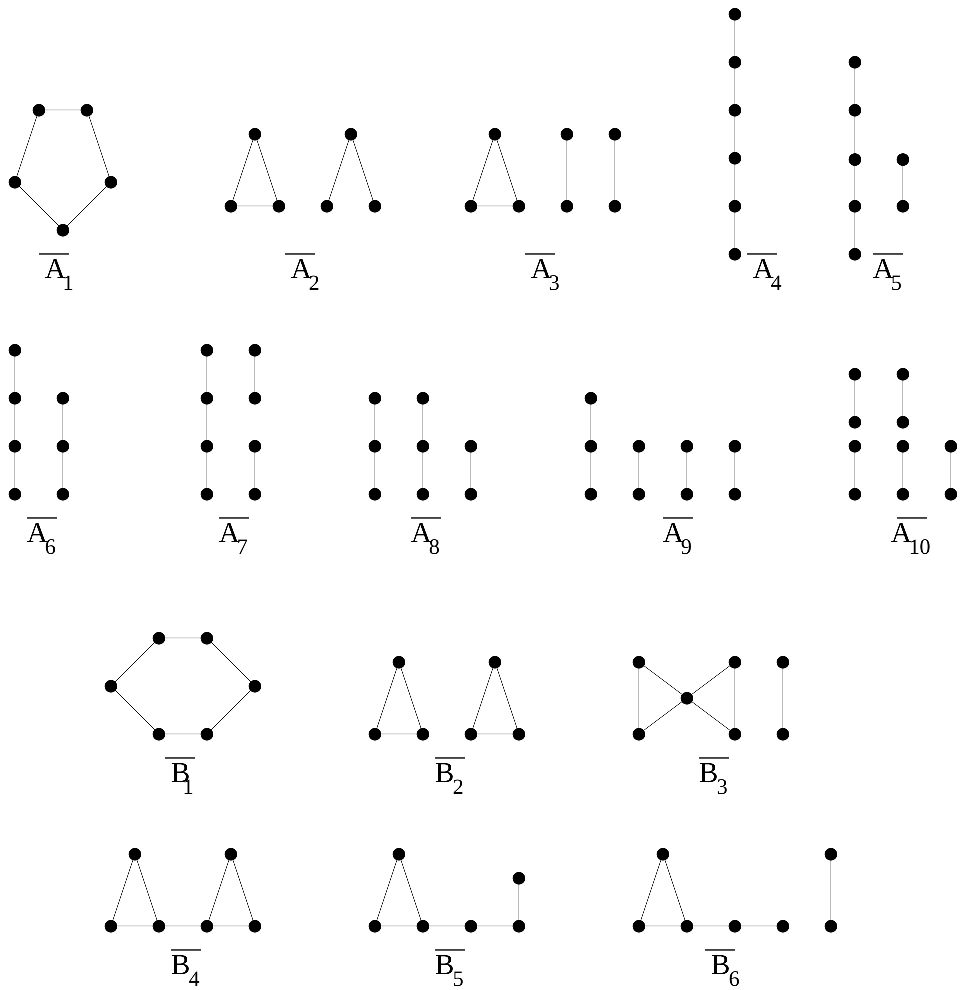

Figure 2: Complements of graphs $A_1 - A_{10}$, $B_1 - B_6$ (the 3-blocks over $\mathbb{R}$)

We now characterize the graphs having order ≤ 2 over $\mathbb{R}$. The result relies essentially on a graph-theoretic result concerning the characterization in terms of forbidden induced subgraphs of the graphs in the classes $\mathcal{G}_i$ $(i = 1, 2, 3, 4)$ and their clique sums. We first formulate this graph-theoretic result whose proof, in view of its length, is delayed till Section 5.

Theorem 9. *The following assertions are equivalent for a graph G.*

(i) *G does not contain as an induced subgraph a circuit C_n ($n \geq 5$) or any of the graphs A_2-A_{10} and B_1-B_6 (cf. Figure 2).*

(ii) *G is a clique sum of a set of graphs belonging to $\bigcup_{i=1}^{4} \mathcal{G}_i$ (cf. Figure 1).*

We now formulate our characterization for the graphs having real order ≤ 2.

Theorem 10. *The following assertions are equivalent for a graph G.*

(i) $\mathrm{ord}_{\mathbb{R}}(G) \leq 2$.

(ii) *G does not contain as an induced subgraph a circuit C_n ($n \geq 5$) or any of the graphs A_2-A_{10} and B_1-B_6 (cf. Figure 2).*

(iii) *G is a clique sum of a set of graphs belonging to $\bigcup_{i=1}^{4} \mathcal{G}_i$ (cf. Figure 1).*

Proof. The implication (i) $\Longrightarrow$ (ii) follows from the fact that the graphs C_n ($n \geq 5$), $A_1 - A_{10}$, $B_1 - B_6$ all have order ≥ 3 (for this, it suffices to exhibit for each of them a 3-dimensional extremal orthogonal representation; cf. [AHMR88]). The implication (ii) $\Longrightarrow$ (iii) holds by Theorem 9, while (iii) $\Longrightarrow$ (i) follows from Propositions 4 and 8. ∎

The result from Theorem 10 can be seen as an analogue and generalization of the corresponding characterization for graphs of order 1, which states that the following assertions are equivalent for a graph G: (i) $\mathrm{ord}_{\mathbb{R}}(G) = 1$; (ii) G is chordal (i.e., does not contain as an induced subgraph any circuit C_n ($n \geq 4$)); (iii) G can be decomposed as a clique sum of cliques. As we explain in Section 5, our proof for (ii) $\Longrightarrow$ (iii) (that is, the proof of Theorem 9) mimicks the proof given by Schrijver [Sc94] for the corresponding implication in the chordal case; the details in our case are, however, technically more involved.

We now mention some applications of Theorem 10. A first application is that one can test in polynomial time whether a given graph G has order ≤ 2 over the reals. Indeed, it suffices for this to first (i) decompose G into graphs without clique cutsets by means of clique sums and then to (ii) test whether all the graphs produced by step (i) belong to $\bigcup_{i=1}^{4} \mathcal{G}_i$. By the algorithm of Tarjan [Ta85], step (i) can be performed in time $O(nm)$ if G has n nodes and m edges; in particular, one can test in polynomial time if a graph G is chordal. Step (ii) can also be executed in polynomial time, since one can obviously test in polynomial time whether a graph belongs to $\mathcal{G}_i$ ($= 1, 2, 3, 4$).

As another application, we can derive the classification of the 3-blocks over the reals, which was obtained by Agler et al. [AHMR88]. The only fact from [AHMR88] that we have used concerning the graphs $A_1 - A_{10}$, $B_1 - B_6$ is that they have order ≥ 3 (which easily implies that they are 3-blocks). But, we obtain 'for free' the hard part, which consists of showing that $A_1 - A_{10}$, $B_1 - B_6$ are the *only* 3-blocks.

Theorem 11. *In the real case, the 3-blocks are the graphs $A_1 - A_{10}$ and $B_1 - B_6$.*

PROOF. If G is a 3-block then, by Theorem 10, G must contain one of the graphs $A_1 - A_{10}$, $B_1 - B_6$ as an induced subgraph and, thus, G is equal to it (by the definition of a block). ∎

As another application of Theorem 10, we can characterize the graphs G whose powers $G^{(m)}$ all have order ≤ 2 over $\mathbb{R}$.

Theorem 12. *The following assertions are equivalent for a graph G.*

(o) $\mathrm{ord}_{\mathbb{R}}(G^{(m)}) \leq 2$ *for every integer* $m \geq 1$.

(i) $\mathrm{ord}_{\mathbb{R}}(G^{(2)}) \leq 2$.

(ii) *G does not contain as an induced subgraph a circuit C_n ($n \geq 5$) or any of the graphs A_4, B_1, D_1, D_2, D_3 (cf. Figures 2 and 3).*

(iii) *G is a clique sum of a set of graphs belonging to the class $\mathcal{G}_4$ (cf. Figure 1).*

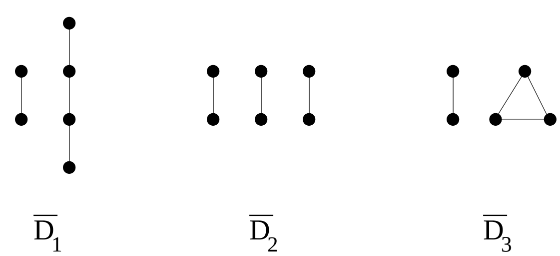

Figure 3: Complements of graphs D_1, D_2, and D_3

The next lemma will be used in the proof of Theorem 12 and later as well.

Lemma 13.

(i) *Let H be a chordal graph that does not contain an induced path of length 3 and with stability number $\alpha(H) = 2$. Then, its node set can be partitioned into $V_0 \cup V_1 \cup V_2$ in such a way that $V_0 \cup V_1$ and $V_0 \cup V_2$ are cliques and there is no edge between V_1 and V_2.*

(ii) *If $G \in \mathcal{G}_1$ does not contain D_1 or D_3 as an induced subgraph, then $G \in \mathcal{G}_4$.*

PROOF. (i) As H is chordal and is not a clique, there exists a clique cutset K in H. Hence, the node set V_H of H can be partitioned into $N_1 \cup N_2 \cup K$, in such a way that there is no edge between N_1 and N_2. Moreover, both N_1, N_2 are cliques (since $\alpha(H) = 2$). For $a = 1, 2$, set

$$K_a := \{k \in K \mid ik \notin E \ \text{ for some } i \in N_a\}.$$

Then, $K_1 \neq \emptyset \Longrightarrow K_2 = \emptyset$, a node $k \in K_1$ is not adjacent to any node of N_1 (else, one would find a path of length 3 in H), and $k \in K_1$ is adjacent to all nodes in N_2 (since $\alpha(H) = 2$). Therefore, we can assume that $K_2 = \emptyset$ and, then,

$$V_H = N_1 \cup (K \setminus K_1) \cup (K_1 \cup N_2),$$

where the sets $N_1 \cup (K \setminus K_1)$ and $(K \setminus K_1) \cup K_1 \cup N_2$ are cliques and there is no edge between the sets N_1 and $K_1 \cup N_2$. Thus, (i) holds (setting $V_0 := K \setminus K_1$, $V_1 := N_1$, $V_2 := K_1 \cup N_2$).
(ii) Let $G \in \mathcal{G}_1$ and let H denote the chordal part in G (cf. Figure 1 (1a)). If G does not contain D_1 or D_3 as an induced subgraph, then H does not contain an induced path of length 3 and $\alpha(H) \leq 2$. We may assume that H is not a clique (else we are done). We now deduce using (i) that G has indeed the form of a graph in $\mathcal{G}_4$. ∎

PROOF OF THEOREM 12. (o) $\Longrightarrow$ (i) is obvious. The implication (i) $\Longrightarrow$ (ii) follows from the corresponding implication in Theorem 10; indeed, the graphs $H := D_1, D_2, D_3$ are forbidden as induced subgraphs of G since $H^{(2)}$ contains A_6, A_8, A_2, respectively. Similarly, (ii) $\Longrightarrow$ (iii) follows from the corresponding implication in Theorem 10. Indeed, each of the graphs $A_1 - A_{10}$, $B_1 - B_6$ contains one of A_4, B_1, D_1, D_2, D_3 as an induced subgraph. Hence, under assumption (ii), we know that G is a clique sum of a family of graphs belonging to $\bigcup_{i=1}^4 \mathcal{G}_i$. In order to conclude the proof, it suffices now to verify that a graph $G \in \bigcup_{i=1}^3 \mathcal{G}_i$ not containing D_1, D_2, D_3 necessarily belongs to $\mathcal{G}_4$. This is obvious for the classes $\mathcal{G}_2$ and $\mathcal{G}_3$ and Lemma 13 (ii) settles the case when $G \in \mathcal{G}_1$. ∎

4 Graphs of order 2: The complex case

In this section we characterize the graphs having order ≤ 2 over $\mathbb{C}$. As in the real case, we begin with exhibiting some basic classes of graphs having order ≤ 2 over $\mathbb{C}$ as well as some examples of graphs with order ≥ 3. We introduce a new class $\mathcal{G}_5$ consisting of the graphs having the form shown in Figure 4.

Proposition 14. *Every graph $G \in \mathcal{G}_4 \cup \mathcal{G}_5$ satisfies:* $\mathrm{ord}_{\mathbb{C}}(G) \leq 2$.

PROOF. McCullough (Prop. 2.6, [McC88]) has shown that $\mathrm{ord}_{\mathbb{C}}(G) \leq 2$ for $G \in \mathcal{G}_4$. Now, if $G \in \mathcal{G}_5$, then $\mathrm{ord}_{\mathbb{C}}(G) \leq 2$, by (4). ∎

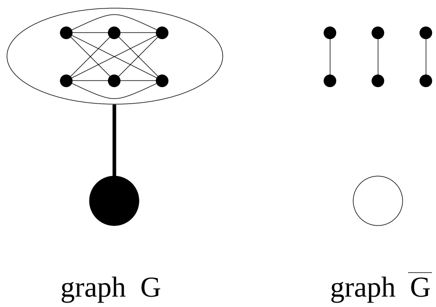

Figure 4: Class $\mathcal{G}_5$

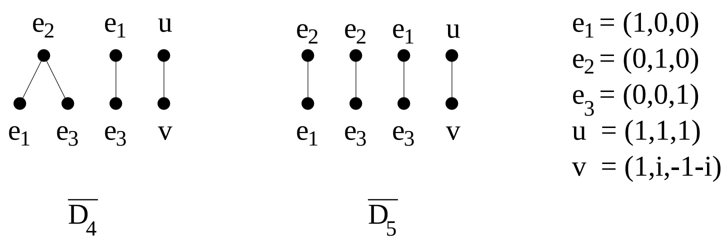

Figure 5: Complements of graphs D_4 and D_5

It is shown in [McC88] that $\mathrm{ord}_{\mathbb{C}}(G) \geq 3$ if G is one of the graphs C_n $(n \geq 5)$, A_4, B_1 (cf. Figure 2), D_1, D_3 (cf. Figure 3). We observe that the graphs D_4 and D_5 whose complements are shown in Figure 5 also have order ≥ 3. (The vector assignments indicated there provide a 3-dimensional extremal orthogonal representation.) In fact, the graphs $A_4, B_1, D_1, D_3, D_4, D_5$ all have order equal to 3 (by (4)). Therefore, the graphs C_n $(n \geq 5)$, $A_4, B_1, D_1, D_3, D_4, D_5$ are forbidden induced subgraphs for the class of graphs having order ≤ 2 over $\mathbb{C}$. As we now see, there are no other minimal forbidden induced subgraphs. The proof of this result relies again on a decomposition result, which follows quite easily from Theorem 9.

Theorem 15. *The following assertions are equivalent for a graph G.*

(i) $\mathrm{ord}_{\mathbb{C}}(G) \leq 2$.

(ii) *G does not contain as an induced subgraph any of the graphs C_n $(n \geq 5)$, A_4, B_1, D_1, D_3, D_4, and D_5.*

(iii) *G is a clique sum of a set of graphs belonging to $\mathcal{G}_4 \cup \mathcal{G}_5$.*

Proof. The implications (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (i) are clear. We now verify the implication (ii) $\Longrightarrow$ (iii). For this, let G be a graph satisfying Theorem 15 (ii). Then, G satisfies the condition (ii) from Theorem 10 (since the graphs A_2, A_3, B_2-B_6 all contain D_3; while A_5, A_6, A_7 contain D_1; A_8, A_9 contain D_4; and finally A_{10} contains D_5). Therefore, by Theorem 10, G is a clique sum of a set of graphs belonging to $\bigcup_{i=1}^{4} \mathcal{G}_i$. In order to conclude the proof, it suffices to verify that a graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ satisfying Theorem 15 (ii) belongs, in fact, to $\mathcal{G}_5$. This is easy to see when $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ and Lemma 13 (ii) settles the case when $G \in \mathcal{G}_1$. ∎

It follows from Theorem 15 that the graphs having order ≤ 2 over $\mathbb{C}$ can be recognized in polynomial time. We can also derive from Theorem 15 the classification of the 3-blocks over $\mathbb{C}$.

Corollary 16. *The 3-blocks over the field $\mathbb{C}$ are the graphs C_5, B_1, A_4, D_1, D_3, D_4, and D_5.* ∎

As another application of Theorem 15, we obtain the following result of McCullough [McC88, McC93] characterizing the graphs G whose powers $G^{(m)}$ all have order ≤ 2 over $\mathbb{C}$. It turns out that we find the same graphs as in the real case.

Theorem 17. *The following assertions are equivalent for a graph G.*

(o) $\mathrm{ord}_{\mathbb{C}}(G^{(m)}) \leq 2$ *for every integer* $m \geq 1$.

(i) $\mathrm{ord}_{\mathbb{C}}(G^{(2)}) \leq 2$.

(ii) *G does not contain as an induced subgraph a circuit C_n ($n \geq 5$) or any of the graphs A_4, B_1, D_1, D_2, D_3 (cf. Figures 2 and 3).*

(iii) *G is a clique sum of a set of graphs belonging to the class $\mathcal{G}_4$ (cf. Figure 1).*

Proof. The implications (o) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (o) are clear. The implication (i) $\Longrightarrow$ (ii) follows from Theorem 15 and the fact that $D_2^{(2)}$ contains D_4. We now verify the implication (ii) $\Longrightarrow$ (iii). For this, let G satisfy Theorem 17 (ii); then, G satisfies Theorem 15 (ii) and, thus, is a clique sum of a set of graphs belonging to $\mathcal{G}_4 \cup \mathcal{G}_5$. It suffices now to note that a graph belonging to $\mathcal{G}_5$ and satisfying Theorem 17 (ii) belongs, in fact, to $\mathcal{G}_4$. ∎

McCullough [McC88, McC93] has given an additional equivalent property for the graphs satisfying Theorem 17 (ii) (called by him *2-chordal*), in terms of existence of a certain linear ordering of the nodes. Recall that chordal graphs are characterized by the fact that they have a *perfect elimination ordering*; that is, an ordering $u_1, \ldots, u_n$ of the nodes such that, for $i = 1, \ldots, n-1$, u_i is simplicial in the graph $G[\{u_i, \ldots, u_n\}]$ (a node being simplicial if the set of nodes to which it is adjacent is a clique). McCullough introduces a notion of 'simplicial pair of nodes' (extending that of simplicial node) and defines accordingly a certain ordering of the nodes, whose existence characterizes 2-chordal graphs. Note, however, that the 'hard' part in his proof lies also in proving the decomposition result via clique sums (the original proof given in [McC88] for this decomposition result was not correct; it was later corrected in [McC93]).

5 Proof of Theorem 9

This section is devoted to the proof of Theorem 9. The implication (ii) $\Longrightarrow$ (i) follows from the fact that the graphs C_n $(n \geq 5)$, $A_2 - A_{10}$, $B_1 - B_6$ have no clique cutset and that they cannot occur as an induced subgraph of a graph in $\mathcal{G}_i$ $(i = 1, 2, 3, 4)$.

We now turn to the proof of the reverse implication: (i) $\Longrightarrow$ (ii). As mentioned earlier, the starting point of our proof was inspired by the proof given in [Sc94] for the following result of Dirac [Di61]: Every chordal graph G which is not a clique has a clique cutset. The latter result can be shown in the following manner.

As G is not a clique, there exists a node u which is not adjacent to all nodes in V. Let $S \subseteq V$ be a maximal subset of V containing u such that $G[S]$ is connected and the set

$$N := \{i \in V \setminus S \mid i \text{ is adjacent to some node in } S\}$$

is strictly contained in $V \setminus S$. Thus, setting $\overline{N} := V \setminus (S \cup N)$, we have partitioned V into

$$V = S \cup N \cup \overline{N},$$

where $G[S]$ is connected, $\overline{N} \neq \emptyset$, and there is no edge between the sets $\overline{N}$ and S. Moreover, it follows immediately from the maximality assumption on S that

(1) every node of N is adjacent to every node of $\overline{N}$.

It now follows that N is a clique and, thus, a clique cutset in G. Indeed, suppose that i, j are two non adjacent nodes in N; let $s \in S$ be adjacent to i, let $t \in S$ be adjacent to j, and let $\overline{n} \in \overline{N}$. Then, considering a path from s to t in $G[S]$ together with edges $\overline{n}i$, $\overline{n}j$, we obtain an induced circuit of length ≥ 4 in G, contradicting the assumption that G is chordal.

We now return to the proof of Theorem 9 (i) $\Longrightarrow$ (ii). For this, we let G be a graph satisfying condition (i) (i.e., G does not contain C_n $(n \geq 5)$, $A_2 - A_{10}$, $B_1 - B_6$ as an induced subgraph) and we assume that G cannot be decomposed as a clique sum (i.e., G has no clique cutset). We show that G belongs to one of the classes $\mathcal{G}_i$, $i = 1, 2, 3, 4$. We may obviously suppose that G is not a clique. Then, in the same manner as above, we can partition the node set V into

$$V = S \cup N \cup \overline{N},$$

where $G[S]$ is connected, $\overline{N} \neq \emptyset$, there is no edge between the sets $\overline{N}$ and S, S is not a clique, and relation (1) holds. The rest of the proof consists in a detailed analysis of the structure of the sets S, N and $\overline{N}$, so that we can finally reach the conclusion that G has indeed the form of a graph in $\bigcup_{i=1}^{4} \mathcal{G}_i$.

5.1 Preliminary results and sketch of proof

We group here a number of preliminary results on the structure of G which will lead to several distinct cases that have to be considered. In what follows, we let $\overline{n}$ denote a given element of $\overline{N}$. For $s \in S$, we set

$$N(s) := \{i \in N \mid is \in E\}.$$

Claim 2. *If st is an edge in S such that $N(s) \cup N(t)$ is not a clique, then $N(s) \subseteq N(t)$ or $N(t) \subseteq N(s)$.*

Proof. Assume that st is an edge in S and that $N(s) \setminus N(t)$, $N(t) \setminus N(s)$ are both non empty; we show that $N(s) \cup N(t)$ is a clique. For this, let $i \in N(s) \setminus N(t)$ and $j \in N(t) \setminus N(s)$; then, $ij \in E$ (else, $(\overline{n}, i, s, t, j)$ would be an induced C_5). Let i' be another node in $N(s) \setminus N(t)$; then, $ii' \in E$ (else, we find B_5 on $\{s, t, i, i', j, \overline{n}\}$). Let $k \in N(s) \cap N(t)$; then, $ki \in E$ (else, we find A_4 or B_1 on $\{\overline{n}, s, t, i, j, k\}$ depending whether $kj \in E$). Finally, if $k, k' \in N(s) \cap N(t)$, then $kk' \in E$ (else, we find A_5 on $\{s, t, k, k', i, j, \overline{n}\}$). ∎

Claim 3. *If $i, j \in N$ are two non adjacent nodes in N, then there exists $s \in S$ which is adjacent to both i and j.*

Proof. Let $i, j \in N$ be non adjacent and assume that no $s \in S$ is adjacent to both i, j. There exist $s, t \in S$ such that $si, tj \in E$ and $sj, ti \notin E$. Consider a shortest path P in S from s to t. Then, this path P together with the edges $si, i\overline{n}, \overline{n}j, jt$ yields a circuit of length ≥ 5 in G, a contradiction. ∎

As a consequence of Claim 3, we obtain that

(4) $G[\overline{N}]$ is chordal.

Indeed, suppose that $G[\overline{N}]$ contains an induced C_4; let $i, j \in N$ be nonadjacent, and let $s \in S$ be adjacent to i and j. Then, we find B_3 on the nodes of C_4 and i, j, s, a contradiction.

Claim 5. *Let $I \subseteq N$ be a stable set in N of cardinality $|I| \geq 3$. Then, there exists a unique node $s \in S$ which is adjacent to all nodes in I.*

PROOF. We proceed by induction on $|I| \geq 3$. Suppose first that $|I| = 3$, $I = \{i, j, k\}$. If there exists no node in S adjacent to i, j, k then, by Claims 3 and 2, there exist pairwise nonadjacent nodes $r, s, t \in S$ such that $ri, rj, si, sk, tj, tk \in E$ and $rk, sj, ti \notin E$; this gives a circuit C_6 in G, a contradiction. Now, if s, t are two distinct nodes in S adjacent to i, j and k, then we find A_2 or B_2 on $\{i, j, k, s, t, \overline{n}\}$ depending whether s, t are adjacent or not. Hence, the result holds when $|I| = 3$. Suppose now that $I = \{i_1, \ldots, i_p\}$ with $p \geq 4$ and that no node of S is adjacent to all elements of I. By the induction assumption, we may assume that, for every $j = 1, \ldots, p$, there exists $s_j \in S$ adjacent to all nodes in $I \setminus \{i_j\}$; then, the subgraph of G induced by nodes $\overline{n}, s_1, s_p, i_{p-2}, i_{p-1}, i_p$ is B_4. Hence, there exists $s \in S$ adjacent to all nodes in I; unicity follows from the case $|I| = 3$. ∎

Claim 6. *Let $i, j, k \in N$ be distinct nodes such that $G[\{i, j, k\}]$ has exactly one edge. Then, there exists a node $s \in S$ which is adjacent to i, j, and k.*

PROOF. Suppose the claim does not hold. Say, $ij \in E$ and $ik, jk \notin E$. Then, there exist $s, t \in S$ such that s is adjacent to i, k but not to j and t is adjacent to j, k but not to i. Then, $st \in E$ (else, we find C_5 on s, t, i, j, k) and we find B_1 on $\overline{n}, s, t, i, j, k$. ∎

Claim 7. *Let $I, J \subseteq N$ be distinct maximal stable sets in N. If $I \cap J \neq \emptyset$, then any node $s \in S$ which is adjacent to all elements in I is adjacent to all elements in J.*

PROOF. Suppose not. Let $s \in S$ be adjacent to all elements in I and let $j \in J \setminus I$ such that $sj \notin E$. By maximality of I, there exists $i \in I$ such that $ij \in E$. Let $k \in I \cap J$ and let $t \in S$ be adjacent to i, j, and k (which exists by Claim 6). Then, we find B_4 or B_5 on $\overline{n}, s, t, i, j, k$ (depending whether $st \in E$ or not). ∎

Claim 8. *Let (i, h, j, k) be an induced C_4 in N (i.e., $ih, jh, ik, jk \in E$, $ij, hk \notin E$). Then, any node $s \in S$ which is adjacent to i and j is adjacent to h and k. Moreover, every node $x \in N \setminus \{i, j, h, k\}$ is adjacent to at least three nodes in $\{i, j, h, k\}$.*

PROOF. Let $s \in S$ be adjacent to i and j and suppose that s is not adjacent, say, to k. Let $t \in S$ be adjacent to h and k. Suppose in a first step that $sh \notin E$.

If t is adjacent to both i, j, then we find B_3 or B_6 on the nodes $\overline{n}, s, t, i, j, h, k$ (depending whether s, t are adjacent); if t is adjacent to one of i, j, then we find A_2 or B_5 and, if t is not adjacent to i, j, then we find B_2 or B_4 on the nodes s, t, i, j, h, k. Therefore, we have that $sh \in E$ and, similarly, $ti \in E$. Then, we find A_5 or B_6 on $\{\overline{n}, s, t, i, j, h, k\}$ when $tj \in E$ and we find A_4 when $tj \notin E$ (on $\{s, t, i, j, h, k\}$ if $st \notin E$ and on $\{\overline{n}, s, t, i, j, k\}$ if $st \in E$).

We now prove the second assertion of the claim. For this, consider $x \in N \setminus \{i, j, h, k\}$ such that $xi \notin E$. Let $s \in S$ which is adjacent to x, i, j (which exists by Claim 6); then s is adjacent to h and k. Hence, the subgraph of G induced by $\{\overline{n}, s, x, i, j, h, k\}$ is A_3, A_5, B_3, or B_6 if one of the edges xj, xh, xk is missing. ∎

Corollary 9. *If $\overline{N}$ is not a clique, then $G[N]$ is chordal and there are at least two edges among any three nodes in N.*

Proof. Suppose that $\overline{n}_1, \overline{n}_2$ are two non adjacent nodes in $\overline{N}$. Suppose first that (i, h, j, k) is an induced C_4 in N. Let $s \in S$ be adjacent to i, j, h, k (which exists by Claims 3 and 8); thus, we find A_3 on $\{\overline{n}_1, \overline{n}_2, s, i, j, h, k\}$. This shows that $G[N]$ is chordal. Suppose now that i, j, k are distinct nodes in N having at most one edge among them. Then, there exists $s \in S$ adjacent to i, j, k (by Claims 5 and 6); thus, we find B_2 or A_2 on $\{\overline{n}_1, \overline{n}_2, s, i, j, k\}$. ∎

Let ν denote the largest cardinality of an induced matching in $\overline{G}[N]$, the complementary graph of $G[N]$. Clearly, $\nu \geq 1$ since N is not a clique and $\nu \leq 4$ (for, otherwise, we would find A_{10} in $G[N]$). Moreover, $\nu \geq 2$ means that $G[N]$ contains an induced C_4, i.e., is not chordal. In fact, we can show that

(10) $\quad \nu \leq 3.$

Indeed, suppose that $\nu \geq 4$ and let $\{i_a j_a \mid a = 1, 2, 3, 4\}$ be an induced matching in $\overline{G}[N]$. By Claim 8, there exists a node $s \in S$ which is adjacent to all nodes i_a, j_a, $a = 1, \ldots, 4$. Then, we find A_{10} on $\{\overline{n}, s, i_a, j_a \ (a = 1, 2, 3, 4)\}$.

We can now indicate what is the overal structure of the proof. We will organize our discussion depending on the value of the parameter ν; namely, $\nu = 1, 2, 3$ (by (10)). In the case when $\nu = 1$, i.e., when the graph $G[N]$ is chordal, it will be convenient to consider separately the two cases when $\alpha(N) = 2$ and when $\alpha(N) \geq 3$. (Recall that $\alpha(N)$ denotes the largest cardinality of a stable set in N.)

To summarize, the proof will consist of examining the following disjoint cases:
Case A: $\nu = 1$ and $\alpha(N) = 2$; then, we show that $G \in \mathcal{G}_1 \cup \mathcal{G}_4$.
Case B: $\nu = 1$ and $\alpha(N) \geq 3$; then, we show that $G \in \mathcal{G}_1$.
Case C: $\nu \in \{2, 3\}$; then, we show that $G \in \mathcal{G}_\nu$.

5.2 Case A

We assume here that $G[N]$ is chordal with stability number $\alpha(N) = 2$. Let $K \subseteq N$ be a clique cutset in $G[N]$. Then, N can be partitioned as

$$N = K \cup N_1 \cup N_2$$

where $N_1, N_2 \neq \emptyset$ and there is no edge between the sets N_1 and N_2. Moreover, both N_1 and N_2 are cliques (by the assumption that $\alpha(N) = 2$). We show that G belongs to $\mathcal{G}_1 \cup \mathcal{G}_4$. For convenience, we introduce the following sets:

$$S_1 := \{s \in S \mid s \text{ is adjacent to } N_1 \text{ but not to } N_2\},$$

$$S_2 := \{s \in S \mid s \text{ is adjacent to } N_2 \text{ but not to } N_1\},$$

$$S_{12} := \{s \in S \mid s \text{ is adjacent to } N_1 \text{ and } N_2\}, \quad S_0 := S \setminus (S_1 \cup S_2 \cup S_{12})$$

and, for $a = 1, 2$,

$$K_a := \{k \in K \mid ik \notin E \;\text{ for some } i \in N_a\} \text{ and } K_0 := K \setminus (K_1 \cup K_2).$$

Given a set $A \subseteq V$ and $u \in V \setminus A$, we say that u *is adjacent to* A if u is adjacent to some element in A. Moreover, a path connecting a node of S_1 to a node of S_2 whose set of internal nodes is contained in S_0 is called a *path from* S_1 *to* S_2 *via* S_0. We have:

$$K_1 \cap K_2 = \emptyset; N_1 \cup K_2 \text{ and } N_2 \cup K_1 \text{ are cliques,}$$

since $\alpha(N) = 2$. Moreover,

$$S_{12} \neq \emptyset.$$

Indeed, given $i_1 \in N_1$, $i_2 \in N_2$, there exists (by Claim 3) a node $s \in S$ which is adjacent to i_1 and i_2; thus, $s \in S_{12}$. The following observation will be used repeatedly.

Claim 11. *There does not exist a path from S_1 to S_2 via S_0. Moreover, any path contained in $S_0 \cup S_1 \cup S_2$ is, in fact, contained in $S_0 \cup S_1$ or $S_0 \cup S_2$.*

PROOF. Suppose that there exists an induced path $(s_1, u_1, \ldots, u_p, s_2)$ where $s_1 \in S_1$, $s_2 \in S_2$, and $u_1, \ldots, u_p \in S_0$ $(p \geq 0)$. Let $i_a \in N_a$ be adjacent to s_a, for $a = 1, 2$. Then, $(i_1, s_1, u_1, \ldots, u_p, s_2, i_2, \overline{n})$ is a circuit of length ≥ 5 in G, yielding a contradiction. The second assertion in the claim follows easily. ∎

By definition, every node of S_{12} is adjacent to at least one node in N_1 and in N_2. More strongly, we have:

(12) Every node $s \in S_{12}$ is adjacent to every node in $N_1 \cup N_2 \cup K_1 \cup K_2$.

This follows using Claim 7, since any two non adjacent nodes of N form a maximal stable set in $G[N]$. Indeed, let $s \in S_{12}$ and let $i_1 \in N_1$, $i_2 \in N_2$ be adjacent to s. Then, s is adjacent to every other node $j_1 \in N_1$ since $\{i_1, i_2\}$ and $\{j_1, i_2\}$ are two intersecting maximal stable sets. Moreover, if $k \in K_1$ is not adjacent to some $j_1 \in N_1$, then s is adjacent to k since $\{k, j_1\}$ is a maximal stable set meeting $\{j_1, i_2\}$.

Claim 13. *The graph $G[S_{12} \cup K \cup \overline{N}]$ is chordal.*

Proof. We already know that $G[K \cup \overline{N}]$ is chordal (using (4)); hence, a possible C_4 is necessarily contained in $K \cup S_{12}$ and has at least two nodes in S_{12}. Let $i_1 \in N_1$ and $i_2 \in N_2$. If (i, j, s, t) is an induced C_4 with $i, j \in K$ and $s, t \in S_{12}$, then $i, j \in K_0$ (by (12)) and we find A_5 on $\overline{n}, s, t, i, j, i_1, i_2$. In the case when (i, r, s, t) is an induced C_4 with $i \in K$ and $r, s, t \in S$, then $i \in K_0$ and we find B_6 on $\overline{n}, r, s, t, i, i_1, i_2$. Finally, if (r, s, t, u) is an induced C_4 contained in S_{12}, then we find B_3 on $\overline{n}, r, s, t, u, i_1, i_2$. ∎

Our next objective is to show that $S = S_{12}$, i.e., that the set $T := S_0 \cup S_1 \cup S_2$ is empty. For this, given $s \in T$, set

$$X_s := \{x \in S_{12} \cup N \mid sx \in E\}.$$

Claim 14. *X_s is a clique for every $s \in T$ and $X_s \cup X_t$ is a clique for every edge st in T.*

Proof. Note first that, if x, y are two non adjacent nodes in $S_{12} \cup N$, then one of the following holds: either $x \in S_{12}$, $y \in S_{12} \cup K_0$; or $x \in K_1$, $y \in N_1$ (or the symmetric case: $x \in K_2$, $y \in N_2$); or $x \in N_1$, $y \in N_2$.

Suppose that X_s is not a clique for some $s \in T$ and let x, y be nonadjacent nodes in X_s. If $x \in S_{12}$, then $N(s) \subseteq N(x)$ (by Claim 2, since $N(x)$ is not a clique) and, thus, $y \notin N$ which, in view of the above observation, means that $y \in S_{12}$. But, then, we find B_4 or B_5 on $\{\overline{n}, s, x, y, i_1, i_2\}$ (where $i_1 \in N_1$, $i_2 \in N_2$). We cannot have: $x \in N_1, y \in N_2$; therefore, $x \in K_1$, $y \in N_1$ and, thus, $s \in S_1$. Then, s must be adjacent to any $i_2 \in N_2$ (by Claim 7, since $\{y, i_2\}$ is a maximal stable set in N meeting $\{x, y\}$), contradicting the fact that $s \in S_1$. Hence, X_s is a clique.

Suppose now that $X_s \cup X_t$ is not a clique for some edge st in T. Then, there exist two nonadjacent nodes x, y in $X_s \cup X_t$ with, say, $x \in X_s \setminus X_t$ and $y \in X_t \setminus X_s$. We can assume, e.g., that $s, t \in S_0 \cup S_1$. Again we cannot have: $x \in N_1, y \in N_2$. Hence, given $i_2 \in N_2$, (i_2, x, s, t, y) is an induced C_5 in the case when $x \in S_{12}$ and $y \in S_{12} \cup K_0$; in the case when $x \in K_1$, $y \in N_1$, then $(\overline{n}, x, s, t, y)$ is an induced C_5. ∎

If $T \neq \emptyset$, we let A denote a maximal subset of T for which $G[A]$ is connected and the set $X(A) := \bigcup_{a \in A} X_a$ is a clique.

Claim 15. *There is no edge between the sets A and $T \setminus A$.*

Proof. Assume that $a \in A$ is adjacent to $b \in T \setminus A$. By maximality of A, we deduce that $X(A) \cup X_b$ is not a clique; hence, there exist two nonadjacent nodes $x \in X(A)$ and $y \in X_b$. Let $a_0 \in A$ be adjacent to x. As x, y are not adjacent, we deduce from Claim 14 that $a_0 \neq a$, $a_0 b, a_0 y, bx \notin E$. Let $(a_0, a_1, \ldots, a_p, a)$ be a shortest path connecting a_0 and a in A (possibly $p = 0$ if $a_0 a \in E$). Together with nodes x, y, this yields an induced path P of length ≥ 4 from x to y and whose internal nodes belong to T. Now, by Claim 11, we may assume that all internal nodes of P belong to $S_0 \cup S_1$. Therefore, the path P together with edges $i_2 x, i_2 y$ (resp. with edges $\overline{n}x$, $\overline{n}y$) yields a circuit of length ≥ 6 in G when $x \in S_{12}$ and $y \in S_{12} \cup K_0$ (resp. when $x \in K_1$ and $y \in N_1$). ∎

We can now deduce that

$$T = S_0 \cup S_1 \cup S_0 = \emptyset; \text{ that is, } S = S_{12}.$$

For, if $T \neq \emptyset$, then $A \neq \emptyset$ and $X(A)$ is a clique cutset in G. (To see it, note that there is no edge between A and the sets $T \setminus A$, $(S_{12} \cup N) \setminus X(A)$ and $\overline{N}$. Hence, if we delete the clique $X(A)$ in G, we obtain a graph in which A is disconnected from the rest of the graph.)

Corollary 16. *If $|N_1| = |N_2| = 1$ and $K_1 = K_2 = \emptyset$, then $G \in \mathcal{G}_1$.*

Proof. Indeed, under this assumption, we have that $N_1 = \{i_1\}$, $N_2 = \{i_2\}$ where both i_1 and i_2 are adjacent to all nodes in $V \setminus \{i_1, i_2\} = S \cup K \cup \overline{N}$. As $G[V \setminus \{i_1, i_2\}]$ is chordal (by Claim 13), we obtain that $G \in \mathcal{G}_1$. ∎

From now on, we can assume without loss of generality that the following holds:

(17) $\quad |N_1| \geq 2$, or $K_1 \neq \emptyset$.

Then, $\overline{N}$ is a clique by Corollary 9. Moreover,

S is a clique and every node of S is adjacent to every node of N.

Indeed, it follows from assumption (17) that $\overline{G}[N]$ contains a path (i, j, k) of length 2 (choosing, either $i, k \in N_1$, $j \in N_2$; or $i \in K_1$, $j \in N_1$, $k \in N_2$ where $ij \notin E$). Therefore, if s, t are two non adjacent nodes of S, then we find A_2 on the set $\{i, j, k, s, t, \overline{n}\}$. Hence, S is a clique. Suppose now that $s \in S$ is not adjacent to some node $h \in N$; then, $h \in K_0$ by (12). Let $t \in S$ be adjacent to h. Then, we find A_6 on the set $\{\overline{n}, s, t, h, i, j, k\}$.

Hence, we know the following information about G: The sets S and $\overline{N}$ are cliques, every node of $S \cup \overline{N}$ is adjacent to every node of N, and $G[N]$ is chordal. This implies:

Corollary 18. *If $|S| = |\overline{N}| = 1$, then $G \in \mathcal{G}_1$.* ∎

Henceforth, we can now assume, moreover, that

(19) $\max(|S|, |\overline{N}|) \geq 2.$

This implies that

$$G[N] \text{ does not contain an induced path of length 3.}$$

Indeed, $\overline{G}[S \cup \overline{N}]$ contains a path of length 2 by (19); hence, if $G[N]$ would contain a path of length 3, we would find graph A_6.

Therefore, by Lemma 13 (i), we know that N can be partitioned into

$$N = V_1 \cup V_0 \cup V_2$$

where $V_0 \cup V_1$ and $V_0 \cup V_2$ are cliques and there is no edge between V_1 and V_2. (Namely, $V_0 = K_0$, $V_1 = N_1$ and $V_2 = K_1 \cup N_2$.) We can now conclude that G belongs to class $\mathcal{G}_4$ (with the cliques $V_1, S, V_2, \overline{N}$ forming the outer circuit and V_0 as central clique). This concludes the proof in case A.

5.3 Case B

We assume here that $G[N]$ is chordal with stability number $\alpha(N) \geq 3$. We show that $G \in \mathcal{G}_1$. For this, let I be a maximal stable set in N; $|I| \geq 3$. By Claim 5, there exists an element $s_I \in S$ which is adjacent to all nodes in I. A first easy observation is that

$$|\overline{N}| = 1.$$

Indeed, if $\overline{n}_1, \overline{n}_2 \in \overline{N}$ and $i, j, k \in I$, then we find A_2 or B_2 on $\{\overline{n}_1, \overline{n}_2, s_I, i, j, k\}$. Next, we observe that

$$s_I \text{ is adjacent to every node of } N.$$

Indeed, let $h \in N \setminus I$. If $hi \notin E$ for some $i \in I$, then s_I is adjacent to h by Claim 7. If $hi \in E$ for all $i \in E$, then $s_I h \in E$ for, if not, we find A_2 on $\{\overline{n}, s_I, h, i, j, k\}$ (where $i, j, k \in I$). At this point, we can already conclude that

(20) if $S = \{s_I\}$, then $G \in \mathcal{G}_1$.

Indeed, s_I and $\overline{n}$ are both adjacent to all elements of N and $G[N]$ is chordal. In what follows, we show that the set $T := S \setminus \{s_I\}$ is indeed empty. For this, note first that

(21) every node $s \in T$ is adjacent to at most one node of I.

Indeed, we know from Claim 5 that s is adjacent to at most two nodes of I. If $s \in T$ is adjacent to $i, j \in I$ and if $k \in I \setminus \{i, j\}$, then we find B_4 or B_5 on $\{\overline{n}, s, s_I, i, j, k\}$. For $s \in T$, set

$$X_s := \{x \in N \mid sx \in E\}.$$

Claim 22. *For every $s \in T$, X_s is a clique and, for every edge st in T, $X_s \cup X_t$ is a clique.*

Proof. Suppose that $x, y \in X_s$ are not adjacent where $s \in T$. Let $i, j \in I$ be both non adjacent to s (such i, j exist by (21)). Then, we have that $xi, yi \in E$ (for, if not, we find A_4, B_4 or B_5 on $\{\overline{n}, s, s_I, x, y, i\}$). Similarly, $xj, yj \in E$. Hence, we have found (x, i, y, j) as induced C_4 in $G[N]$, which contradicts our assumption that $G[N]$ is chordal.

Suppose now that $X_s \cup X_t$ is not a clique for some edge st in T; let $x \in X_s \setminus X_t$, $y \in X_t \setminus X_s$ be non adjacent. Then, $(\overline{n}, x, s, t, y)$ is an induced C_5. ∎

Let A be a maximal subset of T such that $G[A]$ is connected and the set $X(A) := \bigcup_{a \in A} X_a$ is a clique. One can verify that

there is no edge between the sets A and $T \setminus A$.

(The proof is similar to that of Claim 15. Namely, if $a \in A$ is adjacent to $b \in T \setminus A$, then we find two non adjacent nodes $x \in X(A)$, $y \in X_b$ and an induced path P from x to y whose internal nodes belong to T. This path P together with edges $\overline{n}x$, $\overline{n}y$ yields an induced circuit of length ≥ 6.)

From this follows that $A = \emptyset$ (otherwise, the clique $X(A) \cup \{s_I\}$ would be a clique cutset in G). Therefore, $T = \emptyset$, which shows that $S = \{s_I\}$ and, thus, $G \in \mathcal{G}_1$ by (20). This concludes the proof in Case B.

5.4 Case C

We assume here that $\nu \in \{2, 3\}$; thus, $G[N]$ is not chordal. By Corollary 9, this implies that

$\overline{N}$ is a clique.

Let $\{i_a j_a \mid a = 1, \ldots, \nu\}$ be an induced matching of maximum cardinality in $\overline{G}[N]$. In view of Claim 8, every node of N is non adjacent to at most one of the i_a, j_a's. This leads us to defining the following sets:

$$I_a := \{i \in N \mid ij_a \notin E\},\ J_a := \{i \in N \mid ii_a \notin E\}$$

for $a = 1, \ldots, \nu$. Thus, $i_a \in J_a$ and $j_a \in I_a$ for $a \leq \nu$. Moreover, set

$$I := \bigcup_{a=1}^{\nu} I_a \cup J_a,\ N_0 := N \setminus I,$$

$$S_I := \{s \in S \mid s \text{ is adjacent to all nodes of } I\}, \quad \text{and } T := S \setminus S_I.$$

Then, $S_I \neq \emptyset$ (by Claim 8), N_0 is a clique and every node of N_0 is adjacent to every node of I. Moreover,

(23) S_I is a clique.

Indeed, if $s, t \in S_I$ are not adjacent, then we find A_3 on $\{\overline{n}, s, t, i_1, j_1, i_2, j_2\}$. We also have:

$$ij \notin E \text{ for every } i \in I_a, j \in J_a, a = 1, \ldots, \nu.$$

For, if not, then we find A_7 on $\{\overline{n}, s, i, j, i_a, j_a, i_b, j_b\}$ where $s \in S_I$ and $b \in \{1, \ldots, \nu\}$, $b \neq a$. The next statement follows from Claim 8:

(24) Every node of T is adjacent to at most one node of $\{i, j\}$ where $i \in I_a, j \in J_a, a = 1, \ldots, \nu$.

Our next objective is to show that $S = S_I$, i.e., $T = \emptyset$. For this, for $s \in T$, set:

$$X_s := \{x \in S_I \cup N \mid sx \in E\}.$$

Claim 25. *X_s is a clique for every $s \in T$ and $X_s \cup X_t$ is a clique for every edge st in T.*

Proof. Suppose that $x, y \in X_s$ are not adjacent for some $s \in T$. Then, by (23) and (24), we have: $x \in S_I$ and $y \in N$. Using Claim 2, we deduce that $N(s) \subseteq N(x)$, which implies that $xy \in E$, a contradiction.

Suppose now that $X_s \cup X_t$ is not a clique for some edge st in T; let $x \in X_s \setminus X_t$, $y \in X_t \setminus X_s$ be non adjacent. If $x, y \in N$ then $(\overline{n}, x, s, t, y)$ is an induced C_5 in G. Hence, $x \in S_I$ and $y \in N_0$. Consider $i_1 \in I_1, j_1 \in J_1$. By (24), we may assume that $si_1 \notin E$. Then, $ti_1 \in E$ (else, (i_1, x, s, t, y) is an induced C_5) and, thus, $tj_1 \notin E$, $sj_1 \in E$. But, then, we have found B_1 on $\{i_1, j_1, x, y, s, t\}$. ∎

Let A be a maximal subset of T for which $G[A]$ is connected and the set $X(A) := \bigcup_{a \in A} X_a$ is a clique.

Claim 26. *There is no edge between the sets A and $T \setminus A$.*

PROOF. Suppose that $a \in A$ is adjacent to $b \in T \setminus A$. By maximality of A, we deduce that $X(A) \cup X_b$ is not a clique; let $x \in X(A)$, $y \in X_b$ be non adjacent and let $a_0 \in A$ be adjacent to x. We know from Claim 25 that $a \neq a_0$ and $a_0y, xa, xb \notin E$. Considering a shortest path in $G[A]$ from a_0 to a, we find an induced path $(x, a_1, \dots, a_p, y)$ where $a_1, \dots, a_p \in T$ and $p \geq 3$. If $x, y \in N$, then this path together with edges $\overline{n}x, \overline{n}y$ yields an induced circuit in G. Hence, $x \in S_I$ and $y \in N_0$. Consider $i_1 \in I_1$, $j_1 \in J_1$. Then, i_1 is adjacent to one of a_1, a_2. (Indeed, i_1 is adjacent to some a_i for, if not, then $(i_1, x, a_1, \dots, a_p, y)$ is an induced circuit. Let $k \geq 1$ be the smallest index such that $i_1a_k \in E$; then, $(i_1, x, a_1, \dots, a_k)$ is an induced circuit which implies that $k \leq 2$.) Similarly, j_1 is adjacent to one of a_1, a_2. Hence, we can assume that $i_1a_1 \in E$ ($\Longrightarrow j_1a_1 \notin E$), $j_1a_2 \in E$ ($\Longrightarrow i_1a_2 \notin E$). Then, we find B_5 on $\{i_1, j_1, x, y, a_1, a_2\}$. ∎

From this follows that

$$T = \emptyset; \text{ that is, } S = S_I.$$

For, if not, then $A \neq \emptyset$ and $X(A)$ would be a clique cutset in G.

Corollary 27. *If $\nu = 3$, or if $\nu = 2$ with $\max(|I_a \cup J_a| : a = 1, 2) \geq 3$, then $G \in \mathcal{G}_\nu$.*

PROOF. By the assumption, we have that $|\overline{N}| = |S| = 1$ and $S \cup N_0$ is a clique (for, otherwise, one would find A_8 or A_9 in G). Moreover, if $\nu = 3$, then $\max(|I_a \cup J_a| : a = 1, 2, 3) = 2$ (else, one finds A_9). This shows that $G \in \mathcal{G}_\nu$. ∎

Therefore, we can now assume that

$$\nu = 2 \text{ and } \max(|I_a \cup J_a| : a = 1, 2) = 2.$$

Set

$$N_0' := \{i \in N_0 \mid is \notin E \text{ for some } s \in S\}.$$

The following holds:

$$si \notin E \text{ for every } s \in S, i \in N_0'.$$

Indeed, suppose that $si \in E$ for some $s \in S$, $i \in N_0'$ and let $t \in S$ such that $ti \notin E$. Then, we find A_7 on $\{\overline{n}, s, t, i, i_1, j_1, i_2, j_2\}$.

Hence, the node set of G can be partitioned into the sets S, $N_0 \setminus N_0'$, $N_0' \cup \overline{N}$, and $I = \{i_1, j_1, i_2, j_2\}$, in such a way that $S_I \cup (N_0 \setminus N_0')$ and $(N_0 \setminus N_0') \cup N_0' \cup \overline{N}$ are cliques and there is no edge between S_I and $N_0' \cup \overline{N}$. Therefore, G belongs to $\mathcal{G}_2$. This concludes the proof in Case C and, thus, of Theorem 9.

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